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Constraint on linear, homogeneous, constitutive relations

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A constraint on linear, homogeneous, constitutive relations is presented. It does not affect the sources and the primitive fields, its effect being solely on the induction fields. As the Maxwell postulates are satisfied without question, its use amounts to a simplification of the description of linear, homogeneous, magnetoelectric media. The constraint is in complete harmony with the covariant character of modern electromagnetic theory.

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I. INTRODUCTION

Bi-isotropic media are the most general, nondiffusive electromagnetic substances with direction-independent properties. During the last ten years, a few hundred papers on time-harmonic fields in linear bi-isotropic media have appeared in optics, electromagnetics, and electronics journals as well as in other forums. Such great interest may be based on the observation the reciprocal bi-isotropic media, more prominently known as chiral media [1–3], are found commonly in nature as well as in the form of synthetic particulate composites. Publications on nonreciprocal bi-isotropic media have also appeared in considerable abundance, chiefly in this decade [4].

Nonreciprocal bi-isotropic media have never been found in nature. No sample has been synthesized, and no plausible method of fabrication has ever been put forward [5]. Speculation that Cr_2O_3, a uniaxial magnetoelectric material, may be nonreciprocal and bi-isotropic at 1000 Hz frequency and 120 K temperature has been recently shown [6] to have sprung from an unsound premise. The so-called Tellegen medium, which is pathologically bi-isotropic, is macroscopically indistinguishable from isotropic dielectric-magnetic media [7]. Finally, nonreciprocal bi-isotropic media have actually been found [8] to be inconsistent with the structure of electromagnetic theory for linear media.

The last conclusion devolves from a uniformity condition obtained by Post [9] for constitutive parameters, arising from the covariance inherent in the Maxwell postulates. The uniformity condition exists in a very wide context. It can be formulated for the most general, linear, homogeneous medium—commonly referred to as *bi-anisotropic* in modern literature. An important consequence of the uniformity condition is that a linear medium can be nonreciprocal if it is bi-anisotropic [10], but not if it is bi-isotropic. Our objective here is to show that Post's uniformity condition for linear, homogeneous, bi-anisotropic media can be obtained in an alternative fashion.

In the following we use four-vector–tensor notation, first for the time-harmonic electromagnetic field phasors, and then for the time-dependent electromagnetic field vectors. Vectors and phasors are printed boldface and tensors are underlined; a Cartesian coordinate system, indexical notation, and the summation convention are used; the subscript j has three values (1, 2, and 3); and the subscripts k, l, m, and n have four values: 0 (for either frequency ω or time t), as well as 1, 2, and 3 [for space $\mathbf{x} \equiv (x_1, x_2, x_3)$].

II. FREQUENCY DOMAIN

The Maxwell postulates involving electric sources may be expressed in compact tensor notation as

$$\delta_k(\omega,\mathbf{x}) G_{kl}(\omega,\mathbf{x}) = C_l(\omega,\mathbf{x}) \quad (k,l=0,1,2,3) , \tag{1}$$

where

$$\delta_0(\omega,\mathbf{x}) = -i\omega , \tag{2a}$$

$$\delta_j(\omega,\mathbf{x}) = \frac{\partial}{\partial x_j} \quad (j=1,2,3) , \tag{2b}$$

are the components of an operator of mixed algebraic and/or differential type, ω being the angular frequency and $i=\sqrt{-1}$. The second-rank antisymmetric tensor $\underline{G}(\omega,\mathbf{x})$ consists of the induction field phasors as per

$$G_{0j}(\omega,\mathbf{x}) = D_j(\omega,\mathbf{x}) \quad (j=1,2,3) , \tag{3a}$$

$$G_{23}(\omega,\mathbf{x}) = H_1(\omega,\mathbf{x}) , \tag{3b}$$

$$G_{31}(\omega,\mathbf{x}) = H_2(\omega,\mathbf{x}) , \tag{3c}$$

$$G_{12}(\omega,\mathbf{x}) = H_3(\omega,\mathbf{x}) , \tag{3d}$$

while the electric charge density phasor $\rho(\omega,\mathbf{x})$ and the electric current density phasor $\mathbf{J}(\omega,\mathbf{x})$ comprise the right side of (1) with

$$C_0(\omega,\mathbf{x}) = \rho(\omega,\mathbf{x}) , \tag{4a}$$

$$C_j(\omega,\mathbf{x}) = J_j(\omega,\mathbf{x}) \quad (j=1,2,3) . \tag{4b}$$

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The remaining two Maxwell postulates are stated as

$$\epsilon_{klnm}\delta_l(\omega,\mathbf{x})F_{nm}(\omega,\mathbf{x})=0 \quad (k,l,n,m=0,1,2,3)\ , \tag{5}$$

where $\underline{\epsilon}$ is the fourth-rank permutation tensor, and the second-rank antisymmetric tensor $\underline{F}(\omega,\mathbf{x})$ consists of the primitive field phasors with

$$F_{j0}(\omega,\mathbf{x})=E_j(\omega,\mathbf{x}) \quad (j=1,2,3)\ , \tag{6a}$$

$$F_{23}(\omega,\mathbf{x})=B_1(\omega,\mathbf{x})\ , \tag{6b}$$

$$F_{31}(\omega,\mathbf{x})=B_2(\omega,\mathbf{x})\ , \tag{6c}$$

$$F_{12}(\omega,\mathbf{x})=B_3(\omega,\mathbf{x})\ . \tag{6d}$$

Linear, homogeneous constitutive relations in the frequency domain take the form

$$G_{kl}(\omega,\mathbf{x})=\tfrac{1}{2}\chi_{klnm}(\omega)F_{nm}(\omega,\mathbf{x})\ , \tag{7}$$

the constitutive properties being contained in the fourth-rank tensor $\underline{\chi}(\omega)$. Because $\underline{F}(\omega,\mathbf{x})$ and $\underline{G}(\omega,\mathbf{x})$ are antisymmetric, the equalities

$$\chi_{klnm}(\omega)=-\chi_{lknm}(\omega)=-\chi_{klmn}(\omega) \tag{8}$$

follow, irrespective of the specific nature of the linear medium under consideration. Thus, there cannot be more than 36 independent elements in the constitutive tensor $\underline{\chi}(\omega)$.

III. CONSTRAINT IN THE FREQUENCY DOMAIN

Suppose now that there is a constitutive tensor $\underline{\chi}'(\omega)$ such that

$$\chi'_{klnm}(\omega)=\chi_{klnm}(\omega)-K(\omega)\epsilon_{klnm}\ , \tag{9}$$

$K(\omega)$ being some arbitrary nonsingular function of ω. Corresponding to $\underline{\chi}'(\omega)$ and $\underline{F}(\omega,\mathbf{x})$, there is a new induction tensor $\underline{G}'(\omega,\mathbf{x})$ defined by

$$G'_{kl}(\omega,\mathbf{x})=\tfrac{1}{2}\chi'_{klnm}(\omega)F_{nm}(\omega,\mathbf{x})\ ; \tag{10}$$

therefore,

$$G'_{kl}(\omega,\mathbf{x})=G_{kl}(\omega,\mathbf{x})-\tfrac{1}{2}K(\omega)\epsilon_{klnm}F_{nm}(\omega,\mathbf{x})\ . \tag{11}$$

But

$$\begin{aligned}\delta_k(\omega,\mathbf{x})G'_{kl}(\omega,\mathbf{x})&=\delta_k(\omega,\mathbf{x})G_{kl}(\omega,\mathbf{x})\\&\quad+\tfrac{1}{2}K(\omega)\delta_k(\omega,\mathbf{x})\epsilon_{lknm}F_{nm}(\omega,\mathbf{x})\\&=C_l(\omega,\mathbf{x})+0\ ,\end{aligned} \tag{12}$$

as a consequence of (1) and (5), after making use of the skew-symmetric nature of $\underline{\epsilon}$.

Both $\underline{G}'(\omega,\mathbf{x})$ and $\underline{G}(\omega,\mathbf{x})$ correspond to the same source and primitive field phasors. Because $\{\underline{G}(\omega,\mathbf{x}),\underline{F}(\omega,\mathbf{x})\}$ and $\{\underline{G}'(\omega,\mathbf{x}),\underline{F}(\omega,\mathbf{x})\}$ satisfy the Maxwell postulates in an identical manner for the same source phasors, there is nothing in the foregoing to indicate a preference for $\underline{\chi}(\omega)$ over $\underline{\chi}'(\omega)$, and vice versa. Therefore, we have the luxury to choose $K(\omega)$ advantageously.

Let $\underline{\chi}(\omega)$ be specified in some manner for a certain linear homogeneous medium. Equation (9) can be manipulated as

$$\epsilon_{klnm}\chi'_{klnm}(\omega)=\epsilon_{klnm}\chi_{klnm}(\omega)-24K(\omega)\ , \tag{13}$$

because $\epsilon_{klnm}\epsilon_{klnm}=24$. Suppose we choose

$$K(\omega)=\frac{\epsilon_{klnm}\chi_{klnm}(\omega)}{24} \tag{14}$$

for all ω; then

$$\epsilon_{klnm}\chi'_{klnm}(\omega)\equiv 0\ , \tag{15}$$

in conformity with Post's uniformity condition [8,9].

Occam's razor, or the principle of parsimony, now enjoins us to entertain the following proposition:

> For every $\underline{\chi}(\omega)$ there exists a $\underline{\chi}'(\omega)$ such that (15) holds for all ω, and the Maxwell postulates for any prescribed source and primitive field phasors are satisfied identically.

In other words, if $\epsilon_{klnm}\chi_{klnm}(\omega)=0$ for all ω, then $K(\omega)\equiv 0$ and $\underline{\chi}'(\omega)\equiv\underline{\chi}(\omega)$. Therefore, we can always agree to choose the frequency-dependent constitutive tensor $\underline{\chi}(\omega)$ of a specific material to satisfy the *constraint*

$$\epsilon_{klnm}\chi_{klnm}(\omega)\equiv 0 \tag{16}$$

at all frequencies. The *advantage* of this constraint is that $\underline{\chi}(\omega)$ then has $35=36-1$ independent elements instead of 36.

The nondissipative and/or the reciprocal natures of a specific material at a certain frequency may further restrict the number of independent elements. There is an important conceptual difference between these latter restrictions and the constraint (16). While considerations of nondissipation and reciprocity are based on the physical nature of a medium, the constraint (16) arises purely as a *mathematical* consequence of the structure of the Maxwell postulates.

Equation (16) may be set in simpler forms, after noting that it implies that

$$\chi_{0123}(\omega)+\chi_{0231}(\omega)+\chi_{0312}(\omega)+\chi_{2301}(\omega)+\chi_{3102}(\omega)+\chi_{1203}(\omega)=0\ . \tag{17}$$

It does not affect the purely dielectric and the purely magnetic properties, but casts its shadow only on the magnetoelectric coupling effects. When the linear constitutive relations are cast in the Boys-Post representation as

$$\mathbf{D}(\omega,\mathbf{x})=\underline{\epsilon}_{\mathrm{BP}}(\omega)\cdot\mathbf{E}(\omega,\mathbf{x})+\underline{\alpha}_{\mathrm{BP}}(\omega)\cdot\mathbf{B}(\omega,\mathbf{x})\ , \tag{18a}$$

$$\mathbf{H}(\omega,\mathbf{x})=\underline{\beta}_{\mathrm{BP}}(\omega)\cdot\mathbf{E}(\omega,\mathbf{x})+\underline{\mu}^{-1}(\omega)\cdot\mathbf{B}(\omega,\mathbf{x})\ , \tag{18b}$$

the constraint may be set down as [6,8]

$$\mathrm{Tr}\{\underline{\alpha}_{\mathrm{BP}}(\omega)\}-\mathrm{Tr}\{\underline{\beta}_{\mathrm{BP}}(\omega)\}=0\ . \tag{18c}$$

Relevant to the Tellegen constitutive relations,

$$\mathbf{D}(\omega,\mathbf{x})=\underline{\epsilon}_T(\omega)\cdot\mathbf{E}(\omega,\mathbf{x})+\underline{\alpha}_T(\omega)\cdot\mathbf{H}(\omega,\mathbf{x})\ , \tag{19a}$$

$$\mathbf{B}(\omega,\mathbf{x})=\underline{\beta}_T(\omega)\cdot\mathbf{E}(\omega,\mathbf{x})+\underline{\mu}(\omega)\cdot\mathbf{H}(\omega,\mathbf{x}) , \qquad (19b)$$

we get [6,10]

$$\mathrm{Tr}\{\underline{\alpha}_T(\omega)\cdot\underline{\mu}^{-1}(\omega)\}+\mathrm{Tr}\{\underline{\mu}^{-1}(\omega)\cdot\underline{\beta}_T(\omega)\}=0 , \qquad (19c)$$

equivalently.

IV. CONSTRAINT IN THE TIME DOMAIN

Time-dependent quantities are obtained from the corresponding phasors using the inverse Fourier transform. Thus, (1) and (5) may be respectively set down as

$$\tilde{\delta}_k(t,\mathbf{x})\tilde{G}_{kl}(t,\mathbf{x})=\tilde{C}_l(t,\mathbf{x}) , \qquad (20)$$

$$\epsilon_{klnm}\tilde{\delta}_l(t,\mathbf{x})\tilde{F}_{nm}(t,\mathbf{x})=0 , \qquad (21)$$

where

$$\tilde{F}_{nm}(t,\mathbf{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_{nm}(\omega,\mathbf{x})e^{-i\omega t}d\omega , \qquad (22)$$

etc., and

$$\tilde{\delta}_0(t,\mathbf{x})=\frac{\partial}{\partial t} , \qquad (23a)$$

$$\tilde{\delta}_j(t,\mathbf{x})=\frac{\partial}{\partial x_j} \quad (j=1,2,3) , \qquad (23b)$$

are the components of a purely differential operator. Corresponding to (7), the constitutive relations of a homogeneous, linear, causal medium are given by

$$\tilde{G}_{kl}(t,\mathbf{x})=\tfrac{1}{2}\int_{-\infty}^{t}\tilde{\chi}_{klnm}(t-\tau)\tilde{F}_{nm}(\tau,\mathbf{x})d\tau , \qquad (24)$$

where

$$\tilde{\chi}_{klnm}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_{klnm}(\omega)e^{-i\omega t}d\omega \qquad (25)$$

must also conform to the Kramers-Kronig relations.

Our decision to always choose only that $\underline{\chi}(\omega)$ which satisfies (16) at all frequencies turns out to be very felicitous, because it follows from (25) that

$$\epsilon_{klnm}\tilde{\chi}_{klnm}(t)\equiv 0 , \qquad (26)$$

for all time. We note, also with satisfaction, that (26) is identical to Post's uniformity condition [9]. Causality, of course, adds the requirement $\tilde{\chi}_{klnm}(t)=0$ for $t<0$.

V. CONCLUDING REMARKS

Constitutive tensors are given *ab initio* to specify the properties of a medium. Hence, constraints on them are desirable to reduce mathematical arbitrariness without sacrificing physical rectitude.

In this paper, (16) and (26) constitute a constraint on the constitutive relations of a linear homogeneous medium. This constraint does not affect the sources and the primitive fields, its effect being solely on the induction fields as per (12) or the inverse Fourier transform of (12). The Maxwell postulates are satisfied without question, so using the constraint amounts to a simplification of the description of homogeneous media. No physically available material is known to violate the presented constraint. Next, the constraint is identical to the uniformity condition derived by Post from general covariance requirements. Finally, the constraint does not affect pure dielectric-magnetic media, but is relevant to linear magnetoelectric media; we hope the presented work shall illuminate the "many erroneous theoretical studies" noted by Turov [11] in the literature on magnetoelectric media.

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